JOURNAL OF APPROXIMATION THEORY 10, 112–122 (1974)

Whittaker Constants, II

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Communicated by I. J. Schoenberg

1. INTRODUCTION

Let $f(z)$ be a function analytic in a neighborhood of 0. The Gončarov constant G is the infimum of the numbers c for which there exists a function f analytic in $|z| < 1, f \not\equiv 0$, and a sequence of points $\{z_n\}_{n=0}^{\infty}$, $|z_n| \leqslant c/(n+1)$, such that $f^{(n)}(z_n) = 0$. The Whittaker constant W is defined similarly for functions of exponential type 1, and with $|z_n| \leqslant c$. In [1], Boas conjectured that $G = W$, and this conjecture was settled in the affirmative by the first author in [2].

Pommiez [3] has studied a problem which bears a striking resemblence to the above. The shift operator $\mathscr{S}$ transforms the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{into} \quad \mathscr{S}f(z) = \sum_{n=1}^{\infty} a_n z^{n-1}.$$

For $k = 0, 1, 2,\ldots$, the kth iterate of $\mathscr{S}$ is given by

$$\mathscr{S}^k f(z) = \sum_{n=k}^{\infty} a_n z^{n-k}.$$

Pommiez defines the Whittaker constant for the shift operator to be the infimum of the numbers c for which there exists a function f analytic in $|z| < 1$, $f \not\equiv 0$, and a sequence of points $\{z_n\}_{n=0}^{\infty}$, $|z_n| \leqslant c$, such that

$\mathscr{S}^k f(z_k) = 0$. He also defines a similar constant for entire functions of order ρ and type τ, with $|z_k| \leqslant c(k+1)^{1/\rho}/(\rho\tau)^{1/\rho}$. In [4], Shaw proved that the two constants for the shift operator were also equal.

In [5], we introduced a class of operators $\mathfrak{A}$ for analytic functions which includes the derivative and shift operators as special cases. For a specific operator $\mathscr{D} \in \mathfrak{A}$, we define a growth measure for analytic functions called E-type, and give a characterization of the behavior of the zeros of f, $\mathscr{D}f$, $\mathscr{D}^2 f$,... for analytic functions of finite E-type. This characterization includes, as special cases, the determination of the Whittaker constant for both the shift and derivative operators. In this paper, we consider a more general setting, in which the growth measure need not be defined in terms of the specific operator $\mathscr{D}$, and investigate the behavior of the zeros of f, $\mathscr{D}f$, $\mathscr{D}^2 f$,... . We obtain a relationship between these zeros and functions of a certain specified growth which allows us to conclude, in the special case of derivatives, that $G = W$; also that equality holds between the two constants in the case of the shift operator.

Let $\{d_n\}_{n=1}^{\infty}$ denote a nondecreasing sequence of positive numbers, and let $\mathscr{D}$ denote the operator which transforms the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{into} \quad \mathscr{D}f(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}.$$

For $k = 0, 1, 2,...$, the kth iterate of $\mathscr{D}$ is given by

$$\mathscr{D}^k f(z) = \sum_{n=k}^{\infty} d_n \cdots d_{n-k+1} a_n z^{n-k} = \sum_{n=k}^{\infty} (e_{n-k}/e_n)\, a_n z^{n-k},$$

where $e_0 = 1$, and $e_n = (d_1 \cdots d_n)^{-1}$, $n = 1, 2,...$. If $d_n = n$, $\mathscr{D}$ corresponds to the ordinary derivative operator, which we denote by D. If $d_n \equiv 1$, $\mathscr{D}$ reduces to the shift operator $\mathscr{S}$.

Set

$$E(z) = \sum_{n=0}^{\infty} e_n z^n,$$

and let $c(E)$ denote the radius of convergence of E. From the monotonicity of $\{d_n\}$ we have

$$c(E) = \lim_{n\to\infty} d_n = \sup_{1 \leqslant n < \infty} d_n .$$

Define the E-type of the function $f(z) = \sum a_n z^n$ to be the number

$$\tau_E(f) = \limsup_{n\to\infty} |a_n/e_n|^{1/n} = \limsup_{n\to\infty} |\mathscr{D}^n f(0)|^{1/n}.$$

If $c(E) < \infty$, we see that

$$\tau_E(f) = c(E)/c(f),$$

where $c(f)$ denotes the radius of convergence of f. If $E(z)$ is entire, E-type is a growth measure introduced by L. Nachbin [6] which can be related to the growth of the maximum modulus of f. In particular, if $d_n = n$, E-type agrees with exponential type and $E(z) = e^z$.

We shall say the operator $\mathscr{D}$ possess a Whittaker constant if there is a number $W(\mathscr{D})$ with the following properties:

(i) $0 < W(\mathscr{D}) < c(E)$;

(ii) If $\tau_E(f) < W(\mathscr{D})$, and each of f, $\mathscr{D}f$, $\mathscr{D}^2 f$,... has a zero in the closed disk $\mathscr{U} = \{z: |z| \leqslant 1\}$, then $f \equiv 0$;

(iii) There exists a function F such that $\tau_E(F) = W(\mathscr{D})$ and each of F, $\mathscr{D}F$, $\mathscr{D}^2F$,... has a zero in $\mathscr{U}$.

In [5], we proved that every operator $\mathscr{D}$ possesses a Whittaker constant, and gave an exact determination for $W(\mathscr{D})$.

For a function f analytic in a neighborhood of 0, let $r_n(f)$ denote the least modulus of a zero of $\mathscr{D}^n f$ ($r_n(f) = \infty$ if $\mathscr{D}^n f$ has no zero), and define the number $h_n(f)$ by

$$h_n(f) = \sup\{r: 0 \text{ does not belong to the convex hull of } \mathscr{D}^n f(|z| \leqslant r)\}.$$

Let $\{R_n\}_{n=1}^{\infty}$ be a second nondecreasing sequence of positive numbers. For a function $f(z) = \sum a_n z^n$, we call the analogous growth measure

$$\tau_R(f) = \limsup_{n\to\infty} |a_n R_1 \cdots R_n|^{1/n}$$

the R-type of f. Additionally, we require that the sequences $\{d_n\}_{n=1}^{\infty}$ and $\{R_n\}_{n=1}^{\infty}$ satisfy

(a) $\{d_{n+1}/d_n\}_{n=1}^{\infty}$ is a nonincreasing sequence with limit 1;

(b) $\lim_{n\to\infty} R_{n+1}/R_n = 1$.

Note that if we take $R_n = (n/\rho\tau)^{1/\rho}$, and f is a function of R-type 1, then f is an entire function of order ρ and type τ.

We shall need the following results from [5].

THEOREM A. *There is a function $\mathscr{W}$ whose E-type is $W(\mathscr{D})$ which has the following properties*:

(a) $\mathscr{W}(0) = 1$ *and* $|\mathscr{D}^k\mathscr{W}(0)| \leqslant (W(\mathscr{D}))^k$ $(k = 1, 2,...)$;

(b) $\max_{0\leqslant k\leqslant 9} |\mathscr{D}^{n+k}\mathscr{W}(0)|/(W(\mathscr{D}))^{n+k} \geqslant 0.04$ $(n = 0, 1, 2,...)$;

(c) *each of* $\mathscr{W}$, $\mathscr{D}\mathscr{W}$, $\mathscr{D}^2\mathscr{W}$,... *has a zero on the circle* $|z| = 1$.

THEOREM B. *If f is an analytic function of E-type* 1 *or less, and f is not a polynomial, then*

$$\limsup_{n\to\infty} h_n(f) \geqslant W(\mathscr{D}).$$

Our principal results are stated below.

THEOREM 1. *Let f be an analytic function with R-type $\tau_R(f) \leqslant 1$, and suppose that f is not a polynomial. Then*

$$\limsup_{n\to\infty} \frac{d_n r_n(f)}{R_n} \geqslant \limsup_{n\to\infty} \frac{d_n h_n(f)}{R_n} \geqslant W(\mathscr{D}). \tag{1.1}$$

Moreover, there is an analytic function F with R-type $\tau_R(F) = 1$ such that equality holds in (1.1).

Note that if we take the sequence $d_n = n$ and $R_n \equiv 1$, we obtain an equivalent form of the statement that $G = W$. Similarly, if we take $R_n = (n/\rho\tau)^{1/\rho}$ and $d_n \equiv 1$, we obtain equality of the two constants for the shift operator.

2. THE SEQUENCE $h_n(f)$

Let $f(z)$ be an analytic function of R-type 1 or less. We begin by establishing the inequality (1.1).

Suppose first that f is of R-type strictly less than 1. For each positive integer n for which $a_{n+1} \neq 0$, consider the sequence

$$\begin{aligned} f_n(z) &= \frac{\mathscr{D}^n f(R_n z/d_n) - \mathscr{D}^n f(0)}{a_{n+1}R_{n+1}}\, e_n \\ &= \sum_{k=1}^{\infty} \frac{a_{n+k} R_n{}^k e_n e_k}{a_{n+1} d_n{}^k e_{n+k} R_{n+1}}\, z^k. \end{aligned}$$

Note that $\tau_R(f_n) = R_n \tau_R(f)/d_n$, and, therefore,

$$c(f_n) = \frac{d_n(\sup_{n\to\infty} R_n)}{R_n \tau_R(f)} \geqslant \frac{d_n}{\tau_R(f)} \geqslant d_n .$$

Since the R-type of f is strictly less than 1, there is an infinite set S of positive integers such that, if $n \in S$,

$$\left| \frac{a_{n+k} R_1 \cdots R_{n+k}}{a_{n+1} R_1 \cdots R_{n+1}} \right| \leqslant 1 \qquad (k = 0, 1, 2, \ldots).$$

Hence, $n \in S$, we have

$$\begin{aligned} |f_n(z)| &\leqslant \sum_{k=0}^{\infty} \frac{R_n{}^k e_n}{R_{n+1} \cdots R_{n+k} e_{n+k} d_n{}^k} e_k \mid z \mid^k \\ &\leqslant \sum_{k=0}^{\infty} \frac{e_n}{e_{n+k} d_n{}^k} e_k \mid z \mid^k. \end{aligned} \tag{2.1}$$

We will use this estimate to show that the sequence $\{f_n\}_{n \in S}$ is uniformly bounded on compact subsets of $c(E)$.

Note that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{e_n}{e_{n+k} d_n{}^k} e_k \mid z \mid^k &= \sum_{k=0}^{\infty} \frac{d_{n+1} \cdots d_{n+k}}{d_n{}^k} \frac{1}{d_1 \cdots d_k} \mid z \mid^k \\ &= \sum_{k=0}^{\infty} \frac{d_{k+1} \cdots d_{n+k}}{d_1 \cdots d_n} \frac{\mid z \mid^k}{d_n{}^k} \leqslant e_n \sum_{k=0}^{\infty} (d_{n+k})^n \frac{\mid z \mid^k}{d_n{}^k}, \end{aligned}$$

and, hence, the radius of convergence of this series is at least d_n. Let

$$\phi_n(z) = \sum_{k=0}^{\infty} \frac{e_n}{e_{n+k} d_n{}^k} e_k \mid z \mid^k, \qquad \mid z \mid < d_n .$$

Then, for each positive integer k, we have

$$\begin{aligned} \frac{e_n}{e_{n+k} d_n{}^k} &= \frac{d_{n+1} \cdots d_{n+k}}{d_n{}^k} \\ &= \left(\frac{d_{n+1}}{d_n}\right)\left(\frac{d_{n+1}}{d_n} \frac{d_{n+2}}{d_{n+1}}\right) \cdots \left(\frac{d_{n+1}}{d_n} \frac{d_{n+2}}{d_{n+1}} \cdots \frac{d_{n+k}}{d_{n+k-1}}\right) \\ &\leqslant \left(\frac{d_n}{d_{n-1}}\right)\left(\frac{d_n}{d_{n-1}} \frac{d_{n+1}}{d_n}\right) \cdots \left(\frac{d_n}{d_{n-1}} \frac{d_{n+1}}{d_n} \cdots \frac{d_{n+k-1}}{d_{n+k-2}}\right) \\ &= \frac{e_{n-1}}{e_{n+k-1} d_{n-1}^k}, \end{aligned}$$

since the sequence $\{d_{n+1}/d_n\}_{n=1}^{\infty}$ is nonincreasing. It follows that the sequence $\{\phi_k(z)\}_{k=n}^{\infty}$ is nonincreasing for each z in $\mid z \mid < d_n$.

Now let K be a compact subset of $c(E)$. Then we may choose an integer N so large that K is a subset of $\mid z \mid < d_N$. From (2.1) we conclude that

$$\mid f_n(z)\mid \leqslant \phi_n(z) \leqslant \phi_N(z), \qquad z \notin K,$$

for $n > N$. It follows immediately that the sequence $\{f_n\}_{n \in S}$ is uniformly bounded on compact subsets of $c(E)$ and, hence, that there is a subsequence $\{f_{n_m}\}_{m=1}^{\infty}$ which converges uniformly on compact subsets of $c(E)$ to a function $g(z)$. Since $\mathscr{D}^k g(0) = \lim_{m \to \infty} \mathscr{D}^k f_{n_m}(0)$, $k = 0, 1, 2,\ldots$, and

$$\lim_{m \to \infty} \left(\frac{e_m}{e_{m+k} d_m{}^k}\right)^{1/m} = \lim_{m \to \infty} \left(\frac{d_{m+1}}{d_m} \cdots \frac{d_{m+k}}{d_m}\right)^{1/m} = 1,$$

we have $\tau_E(g) \geqslant 1$. Also, $\mathscr{D}g(0) = \lim_{m\to\infty} d_{m+1}R_m/R_{m+1}d_m = 1$, and, hence, g is a nonconstant function.

For $j = 0, 1, 2,...,$ we have $h_0(\mathscr{D}^j f_n) = d_n h_{n+j}(f)/R_n$. Since g is a nonconstant function, we have

$$\lim_{m\to\infty} \frac{d_{n_m} h_{n_m}(f)}{R_{n_m}} = \lim_{m\to\infty} h_0(f_{n_m}) = h_0(g),$$

and

$$\begin{aligned}\lim_{m\to\infty} \frac{d_{n_m+j} h_{n_m+j}(f)}{R_{n_m+j}} &= \lim_{m\to\infty} \frac{d_{n_m} h_{n_m+j}(f)}{R_{n_m}}\\ &= \lim_{m\to\infty} h_j(f_{n_m})\\ &= h_j(g),\end{aligned}$$

for every positive integer j such that $\mathscr{D}^j g$ is nonconstant. If, for some j, $\mathscr{D}^j g$ is constant, let $k + 1$ denote the least such j. Then $\mathscr{D}^k g$ is a polynomial of degree 1, and

$$\limsup_{m\to\infty} \frac{d_{n_m+k} h_{n_m+k}(f)}{R_{n_m+k}} = h_k(g) = +\infty.$$

Therefore,

$$\limsup_{n\to\infty} \frac{d_n h_n(f)}{R_n} = +\infty$$

if g is a polynomial, and

$$\limsup_{n\to\infty} \frac{d_n h_n(f)}{R_n} \geqslant \sup_{0\leqslant j<\infty} h_j(g)$$

otherwise. Since

$$\sup_{0\leqslant j<\infty} h_j(g) \geqslant \limsup_{j\to\infty} h_j(g),$$

and $\tau_E(g) \leqslant 1$, the result follows by applying Theorem B.

If f is of R-type $\tau_R(f) \leqslant 1$, we consider the function

$$f_a(z) = f(az), \qquad 0 < a < 1.$$

Then $\tau_R(f_a) = a\tau_R(f) < 1$, and from the above we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{d_n h_n(f)}{R_n} &= a \limsup_{n\to\infty} \frac{d_n h_n(f_a)}{R_n}\\ &\geqslant aW(\mathscr{D}).\end{aligned}$$

Since the left side is independent of a, we have

$$\limsup_{n\to\infty} \frac{d_n h_n(f)}{R_n} \geqslant W(\mathscr{D}).$$

Finally, we note that

$$r_n(f) \geqslant h_n(f) \qquad (n = 0, 1, 2,\ldots),$$

and this implies

$$\limsup_{n\to\infty} \frac{d_n r_n(f)}{R_n} \geqslant \limsup_{n\to\infty} \frac{d_n h_n(f)}{R_n}.$$

This completes the proof of (1.1).

3. An Example

The function $G(z) = \mathscr{W}(z/W(\mathscr{D}))$, where $\mathscr{W}$ is the function of Theorem A, is of E-type 1 and has the property that each of $G, \mathscr{D}G, \mathscr{D}^2G,\ldots$ has a zero on the circle $|z| = W(\mathscr{D})$. Define the function

$$F(z) = 1 + \sum_{k=1}^{\infty} \frac{\mathscr{D}^k G(0)}{R_1 \cdots R_k} z^k,$$

and note that the R-type of $F(z)$ is

$$\limsup_{k\to\infty} |a_k R_1 \cdots R_k|^{1/k} = \limsup_{k\to\infty} |\mathscr{D}^k G(0)|^{1/k} = 1,$$

since the E-type of G is 1. We shall show that the function $F(z)$ satisfies the inequality

$$\limsup_{n\to\infty} \frac{d_n r_n(F)}{R_n} \leqslant W(\mathscr{D}), \tag{3.1}$$

which will complete the proof of Theorem 1.

LEMMA. *Let* $G(z) = \mathscr{W}(z/W(\mathscr{D}))$, *where* $\mathscr{W}$ *is the function of Theorem* A, *and let*

$$F(z) = 1 + \sum_{k=1}^{\infty} \frac{\mathscr{D}^k G(0)}{R_1 \cdots R_k} z^k.$$

If, for $n \geqslant 1$,

$$\begin{aligned} F_n(z) &= \mathscr{D}^n F(R_n z/d_n)\, e_n R_1 \cdots R_n \\ &= \sum_{k=0}^{\infty} \mathscr{D}^{n+k} G(0) \left\{\frac{R_n{}^k}{R_{n+1} \cdots R_{n+k}}\right\}\left\{\frac{e_n}{e_{n+k} d_n{}^k}\right\} e_k z^k, \end{aligned}$$

then

$$\lim_{n\to\infty} \{F_n(z) - \mathscr{D}^n G(z)\} = 0$$

uniformly on compact subsets of $|z| < c(E)$.

Proof. Suppose $\epsilon > 0$ and $0 < r < c(E)$. Choose r_1 so that $r < r_1 < c(E)$, and choose N so that $d_N > r_1$, and so that

$$(r/r_1)^N \phi_N(z) < \epsilon/4.$$

Choose N_1 so that $N_1 > N$ and so that if $n < N_1$, then

$$\sum_{k=0}^{N_1} \left| \frac{R_n^{\ k}}{R_{n+1} \cdots R_{n+k}} \frac{e_n}{e_{n+k} d_n^{\ k}} - 1 \right| e_k \mid z \mid^k < \frac{\epsilon}{2}.$$

Now suppose that $|z| \leqslant r$ and $n < N$. Then

$$F_n(z) - \mathscr{D}^n G(z) = \sum_{k=0}^{N_1-1} \mathscr{D}^{n+k} G(0) \left[\frac{R_n^{\ k}}{R_{n+1} \cdots R_{n+k}} \frac{e_n}{e_{n+k} d_n^{\ k}} - 1 \right] e_k z^k$$

$$+ \sum_{k=N_1}^{\infty} \mathscr{D}^{n+k} G(0) \left[\frac{R_n^{\ k}}{R_{n+1} \cdots R_{n+k}} \frac{e_n}{e_{n+k} d_n^{\ k}} - 1 \right] e_k z^k.$$

Note that from part (b) of Theorem A, we have

$$| \mathscr{D}^k G(0) | \leqslant 1 \qquad (k = 0, 1, 2, \ldots),$$

and, hence, the absolute value of the first sum does not exceed $\epsilon/2$. The absolute value of the second sum does not exceed

$$\sum_{k=N_1}^{\infty} \frac{R_n^{\ k}}{R_{n+1} \cdots R_{n+k}} \frac{e_n}{e_{n+k} d_n^{\ k}} e_k \mid z \mid^k + \sum_{k=N_1}^{\infty} e_k \mid z \mid^k$$

$$\leqslant \sum_{k=N}^{\infty} \frac{e_n}{e_{n+k} d_n^{\ k}} e_k r^k + \sum_{k=N}^{\infty} e_k r^k$$

$$\leqslant 2 \sum_{k=N}^{\infty} \frac{e_n}{e_{n+k} d_n^{\ k}} e_k r^k$$

$$= 2 \left(\frac{r}{r_1} \right)^N \sum_{k=N}^{\infty} \frac{e_n}{e_{n+k} d_n^{\ k}} e_k \left(\frac{r}{r_1} \right)^{k-N} r_1^{\ k}$$

$$\leqslant 2 \left(\frac{r}{r_1} \right)^N \sum_{k=N}^{\infty} \frac{e_n}{e_{n+k} d_n^{\ k}} e_k r_1^{\ k}$$

$$\leqslant 2 \left(\frac{r}{r_1} \right)^N \phi_n(r_1)$$

$$\leqslant 2 \left(\frac{r}{r_1} \right)^N \phi_N(r_1) < \frac{2\epsilon}{4} = \frac{\epsilon}{2}.$$

This completes the proof of the lemma.

To complete the proof of Theorem 1, suppose that (3.1) is false. Then, for some $\epsilon > 0$, we can find an increasing sequence of positive integers $\{n_m\}_{m=1}^{\infty}$ such that

$$r_0(F_{n_m}) = \frac{d_{n_m} r_{n_m}(F)}{R_{n_m}} \geqslant W(\mathscr{D}) + \epsilon, \tag{3.2}$$

and such that $\{\mathscr{D}^{n_m} G\}$ is uniformly convergent on compact subsets of $|z| < c(E)$ to a function g of E-type 1 or less. The function g is not identically zero, since from Theorem A we have

$$0.04 \leqslant \max_{0 \leqslant k \leqslant 9} |\mathscr{D}^{n+k} G(0)| \leqslant 1 \qquad (n = 0, 1, 2, \ldots).$$

The function G has the property that it and each of $\mathscr{D}G, \mathscr{D}^2 G, \ldots$ has a zero on $|z| = W(\mathscr{D})$. Hence, from Hurwitz' theorem

$$r_0(g) = W(\mathscr{D})$$

and

$$r_{n_m}(g) = W(\mathscr{D}) \qquad (m = 1, 2, \ldots).$$

But the lemma guarantees that the sequence $\{F_{n_m}\}$ also has limit g, and our assumption prevents g from having a zero in the disc $|z| < W(\mathscr{D}) + \epsilon$. This establishes (3.1).

4. Univalence of the Sequence $(\mathscr{D}^k f)_0^{\infty}$

Let $\mathscr{D}_1$ be the operator corresponding to the sequence

$$\{n/(n + 1)\, d_{n+1}\}_1^{\infty}.$$

Note that $\mathscr{D}_1 = D\mathscr{D}D^{-1}$, and that $\mathscr{D}_1$ is well defined; that is, the arbitrary constant introduced by D^{-1} is annihilated by $\mathscr{D}$. Since this sequence is nondecreasing, we see that the operator $\mathscr{D}_1$ possesses a Whittaker constant $W(\mathscr{D}_1)$. Note also that the comparison function for $\mathscr{D}_1$ is

$$E_1(z) = \sum_{k=0}^{\infty} \hat{e}_k z^k,$$

where $\hat{e}_0 = 1$ and

$$\hat{e}_n = \left\{ \prod_{k=1}^{n} \frac{k}{k+1}\, d_{k+1} \right\}^{-1} = \frac{n+1}{d_2 \cdots d_{n+1}} = (n + 1)\, d_1 e_{n+1}\,.$$

Therefore, $E_1(z) = d_1 E'(z)$, E_1 has the same radius of convergence as E, and E_1-type agrees with E-type.

Let $\rho_n(f)$ denote the radius of univalence (with respect to 0) of the function $\mathscr{D}^n f$. We shall make use of the fact that if the convex hull of $f'(|z| \leqslant r)$ does not contain 0, then f is univalent in $|z| \leqslant r$ (this is equivalent to the fact that $\mathscr{R}e f' > 0$ implies univalence of f).

Since the sequence $\{(n+1)/(n+2)d_{n+2}/(n/(n+1))d_{n+1}\}_{n=1}^{\infty}$ is nonincreasing and has limit 1, Theorem 1 implies that if f is of R-type 1 or less then

$$\limsup_{n\to\infty} \frac{n}{n+1} \frac{d_n h_n(f)}{R_n} = \limsup_{n\to\infty} \frac{d_n h_n(f)}{R_n} \geqslant W(\mathscr{D}_1).$$

Since f' is of R-type 1 or less if f is, we apply the theorem to f'. Hence, we have

$$\limsup_{n\to\infty} \frac{d_n h_n(f')}{R_n} \geqslant W(\mathscr{D}_1).$$

Now, $\mathscr{D}_1{}^k = (D\mathscr{D}D^{-1})^k = D\mathscr{D}^k D^{-1}$, and we see that

$$h_n(f') = \sup\{r\colon 0 \text{ does not belong to the convex hull of } D\mathscr{D}^n f(|z| \leqslant r)\}.$$

From our earlier observation, we have

$$\rho_0(\mathscr{D}^n f) = \rho_n(f) \geqslant r,$$

for each r in the set above, and this implies that

$$\rho_n(f) \geqslant h_n(f') \qquad (n = 0, 1, 2, \ldots).$$

Thus,

$$\limsup_{n\to\infty} \frac{d_n \rho_n(f)}{R_n} \geqslant \limsup_{n\to\infty} \frac{d_n h_n(f')}{R_n} \geqslant W(\mathscr{D}_1).$$

Finally, we note from our remarks in the Introduction that there is a function $\mathscr{W}_1$ of E_1-type, and hence E-type, $W(\mathscr{D}_1)$ such that each of $\mathscr{W}_1, \mathscr{D}_1\mathscr{W}_1, \mathscr{D}_1{}^2\mathscr{W}_1, \ldots$ has a zero on the disk $|z| = 1$. By an argument similar to that in Section 3, we can show the existence of a function P of R-type 1 such that

$$\limsup_{n\to\infty} \frac{d_n r_n(P)}{R_n} \leqslant W(\mathscr{D}_1),$$

where $r_n(P)$ denote the least modulus of a zero of $\mathscr{D}_1{}^n P$. Let $F(z)$ denote an indefinite integral of P. Then $\mathscr{D}_1{}^n P = D\mathscr{D}^k F$, and since univalent functions have nonvanishing derivatives, we have

$$\rho_n(F) \leqslant r_n(P).$$

This implies

$$\limsup_{n\to\infty} \frac{d_n \rho_n(F)}{R} \leqslant W(\mathscr{D}_1).$$

Collecting these remarks, we have the following theorem.

THEOREM 2. *Let f be an analytic function of R-type* 1 *or less, and suppose f is not a polynomial. Then*

$$\limsup_{n\to\infty} \frac{d_n \rho_n(f)}{R_n} \geqslant W(\mathscr{D}_1).$$

Moreover, there is an analytic function F of R-type 1 *such that equality holds in the above inequality.*

REFERENCES

1. R. P. BOAS, An upper bound for the Gončarov constant, *Duke Math. J.* **4** (1948).
2. J. B. BUCKHOLTZ, Zeros of successive derivatives of entire functions, *Indian J. Math.* **13** (1971), 83–88.
3. M. POMMIEZ, Sur les zéros des restes successifs des series de Taylor, *Ann. Fac. Sci. Univ. Toulouse* **24** (1960), 77–165.
4. J. K. SHAW, Zeros of Partial Sums and Remainders of Power Series, Dissertation, Univ. of Kentucky, June, 1970.
5. J. D. BUCKHOLTZ AND J. L. FRANK, Whittaker constants, *Proc. London Math. Soc.* **23** (1971), 348–370.
6. L. NACHBIN, An extension of the notion of integral functions of the finite exponential type, *Arais Acad. Brasil Ciencias* **16** (1944), 143–147.